**Partial fraction 2**

1. Write $\frac{3-x}{\left(x^{2}+3\right)\left(x+3\right)}$ in partial fractions.

 Put $\frac{3-x}{\left(x^{2}+3\right)\left(x+3\right)}=\frac{A}{x+3}+\frac{f(x)}{x^{2}+3}$

 $A\left(x^{2}+3\right)+\left(x+3\right)f\left(x\right)=3-x…(1)$

 Put $x=-3$, Then $A=\left[\frac{3-x}{x^{2}+3}\right]\_{x=-3}=\frac{3-(-3)}{(-3)^{2}+3}=\frac{1}{2}$

 $\frac{1}{2}\left(x^{2}+3\right)+\left(x+3\right)f\left(x\right)=3-x$

 $f\left(x\right)=\frac{\left(3-x\right)-\frac{1}{2}\left(x^{2}+3\right)}{x+3}=\frac{2\left(3-x\right)-\left(x^{2}+3\right)}{2\left(x+3\right)}=\frac{1-x}{2}$

 $∴\frac{3-x}{\left(x^{2}+3\right)\left(x+3\right)}=\frac{1}{2\left(x+3\right)}+\frac{1-x}{2\left(x^{2}+3\right)}$

2. Partial fraction: $\frac{2R-4}{\left(R+1\right)\left(2R-1\right)\left(R-2\right)}$

 $\frac{2R-4}{\left(R+1\right)\left(2R-1\right)\left(R-2\right)}=\frac{2}{\left(R+1\right)\left(2R-1\right)}=\frac{\left[\frac{2}{R+1}\right]\_{R=\frac{1}{2}}}{2R-1}+\frac{\left[\frac{2}{2R-1}\right]\_{R=-1}}{R+1}=\frac{\frac{4}{3}}{2R-1}-\frac{\frac{2}{3}}{R+1}$

3. Partial fraction: $\frac{x³-x-5}{(x+2) (x²+1)}$

 Since both $x³-x-5$ and $(x+2) (x²+1)$ are monic and of degree three,

 $\frac{x³-x-5}{(x+2) (x²+1)}≡1+\frac{A}{x+2}+\frac{Bx+C}{x²+1}…(1)$

 $x³-x-5≡\left(x+2\right)\left(x^{2}+1\right)+A\left(x^{2}+1\right)+\left(Bx+C\right)\left(x+2\right)…(2)$

 Put $x=-2$ in (1), $\left(-2\right)^{3}-(-2)-5≡A\left[(-2)^{2}+1\right]$

 $∴A=-\frac{11}{5}…(3)$

 $\left(3\right)\downright \left(2\right), x³-x-5≡\left(x+2\right)\left(x^{2}+1\right)-\frac{11}{5}\left(x^{2}+1\right)+\left(Bx+C\right)\left(x+2\right)$

 $∴Bx+C=\frac{\left( x³-x-5\right)-\left(x+2\right)\left(x^{2}+1\right)+\frac{11}{5}\left(x^{2}+1\right)}{x+2}=\frac{\frac{1}{5}\left(x^{2}-10x-24\right)}{x+2}=\frac{\frac{1}{5}\left(x+2\right)(x-12)}{x+2}=\frac{1}{5}(x-12)$

 $\frac{x³-x-5}{(x+2) (x²+1)}≡1-\frac{\frac{11}{5}}{x+2}+\frac{\frac{1}{5}(x-12)}{x²+1}$

4. Partial fraction: $\frac{1}{\left(x+1\right)^{3}(x+2)}$ and deduce $\frac{1}{\left(x+1\right)^{n}(x+2)}$

 **Method 1**

 (1) $\frac{1}{\left(x+1\right)(x+2)}=\frac{\left(x+2\right)-\left(x+1\right)}{\left(x+1\right)(x+2)}=\frac{1}{x+1}-\frac{1}{x+2}$

 (2) $\frac{1}{\left(x+1\right)^{2}(x+2)}=\frac{1}{x+1}\left[\frac{1}{\left(x+1\right)(x+2)}\right]=\frac{1}{x+1}\left[\frac{1}{x+1}-\frac{1}{x+2}\right]=\frac{1}{\left(x+1\right)^{2}}-\frac{1}{\left(x+1\right)(x+2)}$

 $=\frac{1}{\left(x+1\right)^{2}}-\frac{1}{x+1}+\frac{1}{x+2}$

 (3) $\frac{1}{\left(x+1\right)^{3}(x+2)}=\frac{1}{x+1}\left[\frac{1}{\left(x+1\right)^{2}(x+2)}\right]=\frac{1}{x+1}\left[\frac{1}{\left(x+1\right)^{2}}-\frac{1}{x+1}+\frac{1}{x+2}\right]$

 $=\frac{1}{\left(x+1\right)^{3}}-\frac{1}{\left(x+1\right)^{2}}+\frac{1}{\left(x+1\right)(x+2)}=\frac{1}{\left(x+1\right)^{3}}-\frac{1}{\left(x+1\right)^{2}}+\frac{1}{x+1}-\frac{1}{x+2}$

 **Method 2**

 $\frac{1}{\left(x+1\right)^{3}(x+2)}+\frac{1}{x+2}=\frac{1}{x+2}\left[\left(\frac{1}{x+1}\right)^{3}+1\right]=\frac{1}{x+2}\left[\frac{1}{x+1}+1\right]\left[\frac{1}{\left(x+1\right)^{2}}-\frac{1}{x+1}+1\right]$

 $=\frac{1}{x+2}\left[\frac{x+2}{x+1}\right]\left[\frac{1}{\left(x+1\right)^{2}}-\frac{1}{x+1}+1\right]=\frac{1}{\left(x+1\right)^{3}}-\frac{1}{\left(x+1\right)^{2}}+\frac{1}{x+1}$

 $∴\frac{1}{\left(x+1\right)^{3}(x+2)}=\frac{1}{\left(x+1\right)^{3}}-\frac{1}{\left(x+1\right)^{2}}+\frac{1}{x+1}-\frac{1}{x+2}$

 **Method 3**

 $\frac{1}{\left(x+1\right)^{3}(x+2)}=\frac{\left[\left(x+1\right)^{3}+1\right]-\left(x+1\right)^{3}}{\left(x+1\right)^{3}(x+2)}=\frac{\left[\left(x+1\right)+1\right]\left[\left(x+1\right)^{2}-\left(x+1\right)+1\right]-\left(x+1\right)^{3}}{\left(x+1\right)^{3}(x+2)}$

 $=\frac{\left(x+2\right)\left[\left(x+1\right)^{2}-\left(x+1\right)+1\right]-\left(x+1\right)^{3}}{\left(x+1\right)^{3}(x+2)}=\frac{1}{x+1}-\frac{1}{\left(x+1\right)^{2}}+\frac{1}{\left(x+1\right)^{3}}-\frac{1}{x+2}$

 In general, $\frac{1}{\left(x+1\right)^{n}(x+2)}=\sum\_{k=1}^{n}\frac{\left(-1\right)^{k+1}}{\left(x+1\right)^{n-k+1}}-\frac{1}{x+2}$

**5.** Evaluate$\sum\_{n=1}^{N}\frac{2n^{2}+6n+6}{n\left(n+1\right)\left(n+2\right)(n+3)}$

 $\sum\_{n=1}^{N}\frac{2n^{2}+6n+6}{n\left(n+1\right)\left(n+2\right)(n+3)}=\sum\_{n=1}^{N}\frac{n\left(n+1\right)+\left(n+2\right)(n+3)}{n\left(n+1\right)\left(n+2\right)(n+3)}=\sum\_{n=1}^{N}\left[\frac{1}{\left(n+2\right)(n+3)}+\frac{1}{n\left(n+1\right)}\right]$

 $=\sum\_{n=1}^{N}\left[\frac{\left(n+3\right)-(n+2)}{\left(n+2\right)(n+3)}\right]+\sum\_{n=1}^{N}\left[\frac{\left(n+1\right)-n}{n\left(n+1\right)}\right]=\sum\_{n=1}^{N}\left[\frac{1}{n+2}-\frac{1}{n+3}\right]+\sum\_{n=1}^{N}\left[\frac{1}{n}-\frac{1}{n+1}\right]$

 $=\left[\frac{1}{1+2}-\frac{1}{N+3}\right]+\left[\frac{1}{1}-\frac{1}{N+1}\right]=\frac{4}{3}-\frac{1}{N+1}-\frac{1}{N+3}=\frac{2N(2N+5)}{3\left(N+1\right)(N+3)}$

6. Partial fraction: $\frac{x^{3}}{\left(x+1\right)\left(x^{2}-1\right)^{3}}$

 $\frac{x^{3}}{\left(x+1\right)\left(x^{2}-1\right)^{3}}=\frac{x^{3}}{\left(x+1\right)^{4}\left(x-1\right)^{3}}$

 $\frac{2x}{\left(x+1\right)\left(x-1\right)}=\frac{1}{x+1}+\frac{1}{x-1}$

 Cube: $\frac{8x^{3}}{\left(x+1\right)^{3}\left(x-1\right)^{3}}=\frac{1}{\left(x+1\right)^{3}}+\frac{3}{\left(x+1\right)^{2}\left(x-1\right)}+\frac{3}{\left(x+1\right)}\frac{1}{\left(x-1\right)^{2}}+\frac{1}{\left(x-1\right)^{3}}$

 $\frac{16x^{3}}{\left(x+1\right)^{3}\left(x-1\right)^{3}}$ $=\frac{2}{\left(x+1\right)^{3}}+3\frac{2}{\left(x+1\right)\left(x-1\right)}\left[\frac{1}{x+1}+\frac{1}{x-1}\right]+\frac{2}{\left(x-1\right)^{3}}$

 $=\frac{2}{\left(x+1\right)^{3}}+3\left[\frac{1}{x-1}-\frac{1}{x+1}\right]\left[\frac{1}{x+1}+\frac{1}{x-1}\right]+\frac{2}{\left(x-1\right)^{3}}$

 $=\frac{2}{\left(x+1\right)^{3}}+\frac{3}{\left(x-1\right)^{2}}-\frac{3}{\left(x+1\right)^{2}}+\frac{2}{\left(x-1\right)^{3}}$

 Multiply both sides by $\frac{1}{x+1}$ ,

 $\frac{16x^{3}}{\left(x+1\right)^{4}\left(x-1\right)^{3}}=\frac{1}{x+1}$ $\left[\frac{2}{\left(x+1\right)^{3}}+\frac{3}{\left(x-1\right)^{2}}-\frac{3}{\left(x+1\right)^{2}}+\frac{2}{\left(x-1\right)^{3}}\right]$

 $=\frac{2}{\left(x+1\right)^{4}}-\frac{3}{\left(x+1\right)^{3}}+\frac{3}{\left(x+1\right)\left(x-1\right)^{2}}+\frac{2}{\left(x+1\right)\left(x-1\right)^{3}}$

 $=\frac{2}{\left(x+1\right)^{4}}-\frac{3}{\left(x+1\right)^{3}}+\frac{3}{\left(x+1\right)\left(x-1\right)^{2}}+\frac{2}{\left(x+1\right)(x-1)}\left[\frac{1}{\left(x-1\right)^{2}}\right]$

 $=\frac{2}{\left(x+1\right)^{4}}-\frac{3}{\left(x+1\right)^{3}}+\frac{3}{\left(x+1\right)\left(x-1\right)^{2}}+\left[\frac{1}{x-1}-\frac{1}{x+1}\right]\left[\frac{1}{\left(x-1\right)^{2}}\right]$

 $=\frac{2}{\left(x+1\right)^{4}}-\frac{3}{\left(x+1\right)^{3}}+\frac{2}{\left(x+1\right)\left(x-1\right)^{2}}+\frac{1}{\left(x-1\right)^{3}}$

 $=\frac{2}{\left(x+1\right)^{4}}-\frac{3}{\left(x+1\right)^{3}}+\left[\frac{1}{x-1}-\frac{1}{x+1}\right]\left[\frac{1}{x-1}\right]+\frac{1}{\left(x-1\right)^{3}}$

 $=\frac{2}{\left(x+1\right)^{4}}-\frac{3}{\left(x+1\right)^{3}}+\frac{1}{\left(x-1\right)^{2}}-\frac{1}{\left(x+1\right)(x-1)}+\frac{1}{\left(x-1\right)^{3}}$

 $\frac{32x^{3}}{\left(x+1\right)^{4}\left(x-1\right)^{3}}=\frac{4}{\left(x+1\right)^{4}}-\frac{6}{\left(x+1\right)^{3}}+\frac{2}{\left(x-1\right)^{2}}-\frac{2}{\left(x+1\right)(x-1)}+\frac{2}{\left(x-1\right)^{3}}$

 $=\frac{4}{\left(x+1\right)^{4}}-\frac{6}{\left(x+1\right)^{3}}+\frac{2}{\left(x-1\right)^{2}}-\left[\frac{1}{x-1}-\frac{1}{x+1}\right]+\frac{2}{\left(x-1\right)^{3}}$

 $\frac{x^{3}}{\left(x+1\right)^{4}\left(x-1\right)^{3}}=\frac{1}{32}\left[\frac{4}{\left(x+1\right)^{4}}-\frac{6}{\left(x+1\right)^{3}}+\frac{1}{x+1}+\frac{2}{\left(x-1\right)^{3}}+\frac{2}{\left(x-1\right)^{2}}-\frac{1}{x-1}\right]$

7. Partial fraction: $\frac{1}{\left(x-1\right)^{2}(x+1)}$

 $\frac{1}{\left(x-1\right)^{2}(x+1)}$

 $=\frac{1}{2}\left[\frac{\left(x+1\right)-(x-1)}{\left(x-1\right)^{2}(x+1)}\right]=\frac{1}{2}\left[\frac{1}{\left(x-1\right)^{2}}\right]-\frac{1}{2}\left[\frac{1}{\left(x-1\right)(x+1)}\right]=\frac{1}{2}\left[\frac{1}{\left(x-1\right)^{2}}\right]-\frac{1}{4}\left[\frac{\left(x+1\right)-(x-1)}{\left(x-1\right)(x+1)}\right]$

 $=\frac{\frac{1}{2}}{\left(x-1\right)^{2}}-\frac{\frac{1}{4}}{x-1}+\frac{\frac{1}{4}}{x+1}$

8. Partial fraction: $\frac{\left(2s^{3}-s^{2}\right)}{\left(4s^{2}-4s+5\right)^{2}}$

 $E=\frac{\left(2s^{3}-s^{2}\right)}{\left(4s^{2}-4s+5\right)^{2}}=\frac{\left(8s^{3}-4s^{2}\right)}{4\left(4s^{2}-4s+5\right)^{2}}$

 Using long division, divide $\left(8s^{3}-4s^{2}\right)$ by $\left(4s^{2}-4s+5\right)$, we can get
the quotient = $2s+1$ and remainder = $-6s-5$.

 $∴8s^{3}-4s^{2}=\left(4s^{2}-4s+5\right)\left(2s+1\right)-6s-5$

 Hence $E=\frac{\left(4s^{2}-4s+5\right)\left(2s+1\right)-6s-5}{4\left(4s^{2}-4s+5\right)^{2}}=\frac{2s+1}{4s^{2}-4s+5}+\frac{-6s-5}{4\left(4s^{2}-4s+5\right)^{2}}$

9. Partial fraction: $ \frac{x²+1}{(x-1)²(x+1)}$

 $\frac{x²+1}{(x-1)²(x+1)}=\frac{\left(x-1\right)\left(x+1\right)+2}{(x-1)²(x+1)}=\frac{1}{x-1}+\frac{2}{(x-1)²(x+1)}=\frac{1}{x-1}+\frac{\left(x+1\right)-(x-1)}{(x-1)²(x+1)}$

 $=\frac{1}{x-1}+\frac{1}{(x-1)²}-\frac{1}{\left(x+1\right)(x-1)}=\frac{1}{x-1}+\frac{1}{(x-1)²}-\frac{1}{2}\left[\frac{1}{x-1}-\frac{1}{x+1}\right]=\frac{1}{2}\left(\frac{1}{x-1}\right)+\frac{1}{(x-1)²}+\frac{1}{2}\left(\frac{1}{x-1}\right)$

10. Partial fraction: $\frac{1}{x\left(x^{2}+ 1\right)^{2}}$

 This can be done just by inspection. First observe that:

 $\frac{1}{x\left(x^{2}+ 1\right)}=\frac{\left(x^{2}+ 1\right)-x^{2}}{x\left(x^{2}+ 1\right)}=\frac{1}{x}-\frac{x}{x^{2}+ 1}…(1)$

 $\frac{1}{x\left(x^{2}+ 1\right)^{2}}=\frac{1}{x^{2}+1}\left[\frac{1}{x\left(x^{2}+ 1\right)}\right]=\frac{1}{x^{2}+1}\left[\frac{1}{x}-\frac{x}{x^{2}+ 1}\right]$, by (1)

 $=\frac{1}{x\left(x^{2}+ 1\right)}-\frac{x}{\left(x^{2}+ 1\right)^{2}}$

 $=\frac{1}{x}-\frac{x}{x^{2}+ 1}-\frac{x}{\left(x^{2}+ 1\right)^{2}}$$,by (1)$

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